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Characterizing riddled fractal sets

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Recent studies have revealed that riddled fractal sets, sets whose conventionally defined fractal dimensions are integers, occur commonly in chaotic dynamical systems. We demonstrate that these exotic fractal sets exhibit a sign-singular scaling behavior with nontrivial scaling exponents. The exponents may then be used to characterize the sets. Numerical examples using both a low-dimensional map and a coupled map lattice are given.

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Fat fractals, fractal sets with positive Lebesgue measures, are found in chaotic dynamical systems. A known example of the fat fractal sets is the set of parameter values for which the one-dimensional logistic map exhibits a chaotic attractor [1–3]. Due to the positive Lebesgue measure, the box-counting dimension d_0 for a fat fractal is an integer [3]. To resolve the predicament of assigning integer dimensions for apparently fractal sets and also to better characterize fat fractals, the concept of exterior dimension d_{ex} was introduced [3]. An appealing aspect of the exterior dimension is that it can be related to the uncertainty exponent α [4,3] by $d_{ex}=D-\alpha$, where D is the phase-space dimension. The uncertainty exponent, besides having the advantage of easy numerical access, also has clear physical meaning: it characterizes the scaling behavior of the probability of generating completely different asymptotic behavior of the system upon small perturbations in initial conditions or parameters. Specifically, let $\mathbf{x}_0$ ($\mathbf{p}_0$) be a randomly chosen initial condition (or a set of parameter values) that leads to one type of asymptotic behavior. Then $P(\epsilon)$, the probability that the perturbed initial condition $\mathbf{x}_0+\epsilon$ (or the perturbed set of parameter values $\mathbf{p}_0+\epsilon$) leads to a different asymptotic behavior, decreases as the perturbation ϵ decreases in magnitude, and typically scales with ϵ as $P(\epsilon)\sim\epsilon^{\alpha}$. If $0\leq\alpha<1$, improving the precision with which initial conditions or parameters are specified does not result in an equivalent improvement in the ability to predict the asymptotic behavior correctly [4]. For fat fractal sets with $0<\alpha<1$ (such as the chaotic parameter set in the logistic map example), the exterior dimension assumes a fractional value, which is well suited for quantifying fat fractals.

Recently, an extreme type of fat fractal sets has been identified in chaotic systems with a simple class of symmetry. These sets are the riddled basins [5–7] and the riddled parameter sets [8]. The physical manifestation of such sets is that for an initial condition or a parameter value that leads to a *chaotic attractor,* there are initial conditions or parameter values arbitrarily nearby that lead to other attractors. Thus, the basins of attraction or the parameter sets for the chaotic attractor are riddled with holes that belong to basins or parameter sets of the other attractors. The uncertainty exponents computed for riddled basins or riddled parameter sets are usually very close to zero and are in fact indistinguishable from zero in numerical computations [8]. As such, significant error occurs if one attempts to predict the asymptotic attractor for a given initial condition or a parameter value that is uncertain, and the situation does not improve even if one reduces the uncertainty over many orders of magnitude [5–8]. A direct consequence, from the viewpoint of characterizing riddled basins or riddled parameter sets, is that the exterior dimensions d_{ex} for these fat fractal sets approach to integer values for $\alpha\approx0$ (since $d_{ex}=D-\alpha$).

In this paper, we propose instead to use the so-called cancellation exponent κ to characterize riddled fractal sets with integer exterior dimensions. The cancellation exponent κ was introduced [9] recently to characterize physical quantities that change sign from positive to negative and vice versa on arbitrarily small scales. Physical quantities with nonzero cancellation exponent are said to exhibit the sign-singular scaling behavior [9]. Here, for riddled fractal sets, we demonstrate that a fundamental physical quantity in chaos theory, the maximum Lyapunov exponent λ, exhibits the sign-singular scaling behavior [10]. More importantly, the cancellation exponent is found to possess different fractional values for different riddled fractal sets and, hence, it is suitable for distinguishing and characterizing riddled fractal sets with $\alpha\approx0$ (or $d_{ex}\approx$integer).

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Mathematically, a signed measure that is defined on a set can take on either positive or negative values. This is in contrast to the conventional probability measure of a set X, which is countably additive and assigns only positive value or zero to any subset of X. To define a signed-measure μ, consider a finite one-dimensional interval I. Let $A \in I$ be a subinterval such that $\mu(A) \neq 0$. The measure μ is sign singular if, for any such interval A (no matter how small), there is an interval B contained in A such that $\mu(B)$ has the opposite sign from $\mu(A)$. Hence, the measure μ changes sign on *arbitrarily small* scales. To quantify signed measures, a cancellation exponent κ was introduced in Ref. [9] as the following. Divide the one-dimensional interval I into $N(\epsilon)$ disjoint subintervals I_i, each of length ϵ. Examine the quantity $\Omega(\epsilon) \equiv \sum_{i=1}^{N(\epsilon)} |\mu(I_i)|$. When the number of subintervals $N(\epsilon)$ is small, or equivalently when ϵ is large, we expect $\Omega(\epsilon)$ to be small because each of the $\mu(I_i)$ is small due to the high degree of cancellation within a subinterval I_i. As $N(\epsilon)$ increases, or as ϵ gets smaller, the cancellation of positive and negative contributions of μ in each subinterval I_i is reduced, thereby causing the sum $\Omega(\epsilon)$ to increase. In general, $\Omega(\epsilon)$ scales with ϵ as $\Omega(\epsilon) \sim (1/\epsilon)^{\kappa}$, where $\kappa \geq 0$ is the cancellation exponent. More rigorously, κ can be defined as [9]

$$\kappa = \lim_{\epsilon \to 0} \sup \frac{\ln \Omega(\epsilon)}{\ln 1/\epsilon} . \tag{1}$$

Thus, a larger value of κ corresponds to a higher degree of cancellation of μ, indicating that the measure changes sign on arbitrarily small scales in a significant way. It can be shown that for probability measures or for signed measures with a smooth bounded probability density, $\kappa = 0$ [9]. In order to have a nontrivial κ, oscillation in sign of μ must occur on arbitrarily small scales. It was demonstrated in Ref. [9] that sign-singular measures with $\kappa > 0$ indeed occur in physical systems such as the magnetic field in fast magnetic dynamos, and velocity derivatives and vorticities in high-Reynolds-number fluid turbulence.

To demonstrate that the sign-singular scaling behavior occurs in chaotic systems possessing riddled fat fractal sets, we have carried out a series of numerical experiments for both low- and high-dimensional systems. Our first numerical example is a two-dimensional map that has been shown to exhibit riddled basins [6]. The map is defined in the region given by $0 \leq x \leq 1$ and $y \geq 0$. For $0 \leq y < 1$, the map is given by

$$x_{n+1} = \begin{cases} (1/a)x_n & \text{for } x_n < a \\ (1/b)(x_n - a) & \text{for } x_n > a \end{cases} , \tag{2}$$

$$y_{n+1} = \begin{cases} c y_n & \text{for } x_n < a \\ d y_n & \text{for } x_n > a \end{cases} , \tag{3}$$

where $0 < a < 1$, $b = 1 - a$, $0 < d < 1$, and $c > 1$. Thus the interval $I_0 \equiv [x,y | 0 \leq x \leq 1, y = 0]$ is invariant under iterations of the map and contains a chaotic attractor with a positive Lyapunov exponent $\lambda_{I_0} = a \ln 1/a + b \ln 1/b$. The y dynamics for $0 < y < 1$ involves both expansion and contraction. As a consequence, some initial conditions in the unit square asymptote to the chaotic attractor in I_0, while some other escape the unit square when $y_n \geq 1$. For $y \geq 1$, we assume there is a stable fixed point at $(\bar{x}, \bar{y})(\bar{y} > 1, 0 < \bar{x} < 1)$. In this case, the map can be written as

$$\begin{aligned} x_{n+1} &= \bar{x} + e^{-\lambda_x}(x_n - \bar{x}) , \\ y_{n+1} &= \bar{y} + e^{-\lambda_y}(y_n - \bar{y}) , \end{aligned} \tag{4}$$

where we assume $\lambda_x > 0$ and $\lambda_y > 0$ so that $x_n \to \bar{x}$ and $y_n \to \bar{y}$ as $n \to \infty$. We also assume that $\lambda_x < \lambda_y$ and, hence, for trajectories with $y_n \geq 1$, the asymptotic attractor has a maximum Lyapunov exponent equal to $-\lambda_x$. It has been shown [6] that when $a < a_c \equiv |\ln d| / (\ln c + |\ln d|)$, the basin of the chaotic attractor in I_0 is riddled: for every initial condition in the unit square that asymptotes to this attractor, there are initial conditions arbitrarily nearby that asymptote to the fixed-point attractor at $(\bar{x}, \bar{y})$. We choose $a = 0.25$, $d = 0.8$, $c = 1.8$ (so that $a_c \approx 0.2752 > a$, and $\lambda_{I_0} \approx 0.5623$), and $\lambda_x = 0.6$. The phase-space structure of the map is quite complicated. This can be seen by computing the basins of the attraction for the two attractors. Figure 1(a) shows the basin of

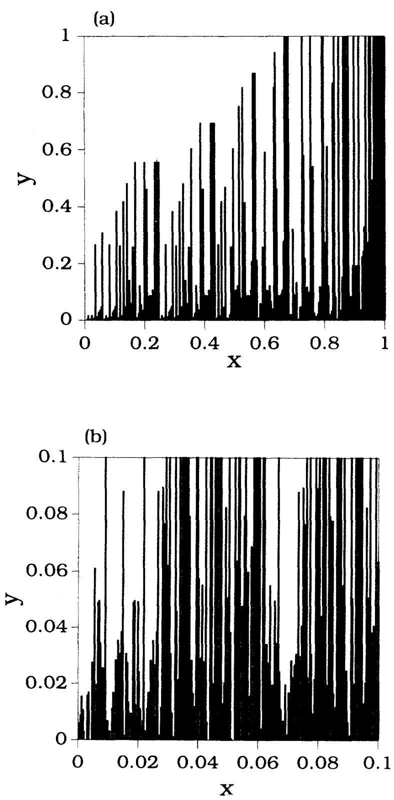

FIG. 1. Basin of the $y = 0$ chaotic attractor for the two-dimensional map (2)–(4) at the following set of parameter values: $a = 0.25$, $d = 0.8$, and $c = 1.8$. (a) In the unit square $0 \leq x, y \leq 1$. (b) A blowup of part of (a) in the region $0 \leq x, y \leq 0.1$.

the $y=0$ chaotic attractor (black dots) in the relevant two-dimensional phase-space region defined by $0 \le x, y \le 1$. The basin of the attractor at $(\bar{x}, \bar{y})$ is represented by blank regions in the plot. It can be seen that for every black dot, there are blank regions arbitrarily nearby, which is typical of a riddled phase space. This behavior persists regardless of the phase-space scale examined, as shown in Fig. 1(b), the basin of the $y=0$ chaotic attractor in the smaller region defined by $0 \le x, y \le 0.1$. The uncertainty exponent in this case is $\alpha \approx 0.003$ [6]. The exterior dimension for the set of initial conditions that asymptote to the chaotic attractor in I_0 is $d_{ex} = 2 - \alpha \approx 1.997$, which is very close to 2, and therefore, d_{ex} is not a good characterizing quantity for the riddled fractal set that is the basin of attraction for the $y=0$ attractor.

To examine the sign scaling behavior of the Lyapunov exponent, we choose 2×10^6 initial conditions uniformly distributed in the interval $I \equiv [x, y | 0 \le x \le 1, y = 0.01]$. Trajectories that have originated from these initial conditions have Lyapunov exponents of either $\lambda = \lambda_{I_0}$ or $-\lambda_x$. Figure 2(a) shows λ versus x, which exhibits a change in the sign of λ that apparently persists on arbitrarily small scales. We examine fluctuations of λ defined by $\Delta\lambda \equiv \lambda - \bar{\lambda}$ for each initial condition, where $\bar{\lambda}$ is the average λ value in the interval. We divide the interval I into $N(\epsilon)$ subintervals with length ϵ, where $N(\epsilon) = [1/\epsilon]$ is the integer part of $1/\epsilon$. For 2×10^6 initial conditions in I, there are $M = [2 \times 10^6 / N(\epsilon)]$ points in each subinterval I_i $[i = 1, \ldots, N(\epsilon)]$. Summing $\Delta\lambda$ values over all M points, we get the quantity $\Delta\lambda(I_i)$. The sum $\Omega(\epsilon) \equiv \sum_{i=1}^{N(\epsilon)} |\Delta\lambda(I_i)|$ can then be computed. Figure 2(b) shows, on a logarithmic scale, the sum versus $1/\epsilon$ for $e^{-8} \le \epsilon \le e^{-3}$, which is fitted by a straight line with slope $\kappa = 0.097 \pm 0.003$ at a 95% confidence level. This clearly indicates a nonzero cancellation exponent and consequently sign-singular scaling behavior in $\Delta\lambda$. Note that the sum starts to deviate from the fitted line at $\ln 1/\epsilon \approx 7.5$. This is a numerical artifact, because, as the interval I_i gets smaller, the number of points contained in each interval also decreases (the total number of points is fixed). Consequently, the degree of cancellation in the whole interval I increases more slowly as even smaller intervals I_i are examined, thereby causing the sum to deviate downwards from the fitted line. At small values of $\ln 1/\epsilon$ in Fig. 2(b), statistical fluctuations of the sum $\Omega(\epsilon)$ become large. The reason is that fewer intervals I_i are involved in the sum as the size of each interval increases. Despite these fluctuations and the deviation of the sum from the fitted line at small values of ϵ, the scaling behavior of the sum with ϵ is robust, and the extraction of the cancellation exponent from the sum is reliable.

Our second numerical example is the following system of N globally coupled circle maps [11,12],

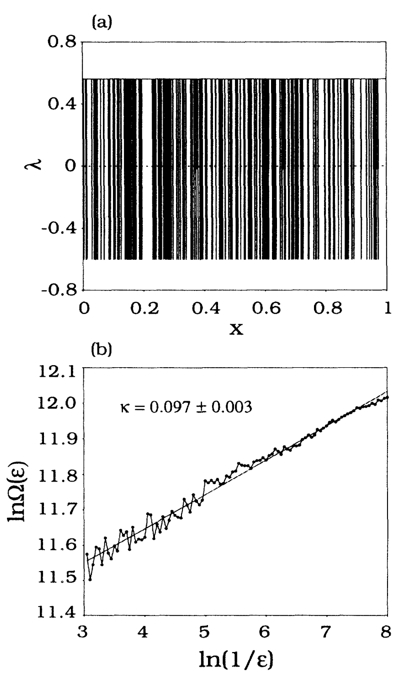

FIG. 2. (a) Lyapunov exponent λ versus x at $y=0.01$ for the two-dimensional map (2)–(4). The parameter setting is the same as in Fig. 1. The uncertainty exponent is $\alpha \approx 0.003$. (b) $\ln\Omega(\epsilon)$ vs $\ln 1/\epsilon$, which gives the cancellation exponent $\kappa = 0.097 \pm 0.003$.

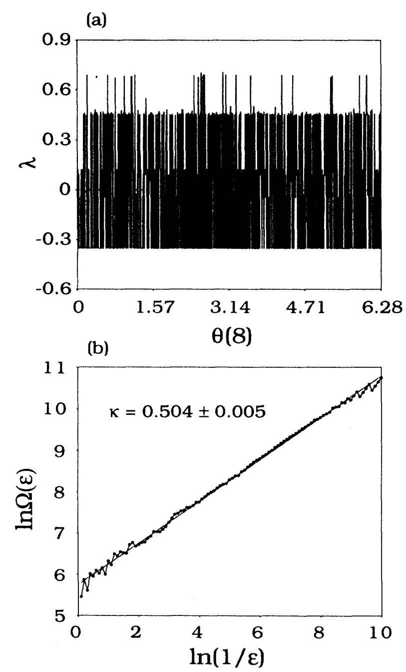

FIG. 3. (a) The maximum Lyapunov exponent λ vs θ at an arbitrary site for the globally coupled circle map lattice Eq. (5) for the following set of parameter values: $k=4$, $\omega=2$, and $\sigma=1.288$. The uncertainty exponent cannot be distinguished from zero in this case. (b) $\ln\Omega(\epsilon)$ vs $\ln 1/\epsilon$, which gives the cancellation exponent $\kappa = 0.504 \pm 0.005$.

$$\theta_{n+1}(i)=\left[\omega+\theta_n(i)+k\sin\theta_n(i)+\frac{\sigma}{N}\sum_{j=1}^{N}\sin\theta_n(j)\right],$$
$$\mathrm{mod}(2\pi),\quad i=1,\ldots,N\ , \qquad (5)$$

where i and n denote discrete spatial site and time, respectively, ω and a are the parameters of the single circle map, and σ is the coupling strength. It has been demonstrated that, in substantial regions of the parameter space, this system exhibits multiple chaotic and nonchaotic attractors in the phase space [12]. The sets of initial conditions in the basins of the chaotic attractors are riddled fat fractals with a near-zero uncertainty exponent (or $d_{ex} \approx D$). Figure 3(a) shows the maximum Lyapunov exponent versus θ on an arbitrary spatial site for $N=20$, $k=4$, $\omega=2$, and $\sigma=1.288$. The uncertainty exponent is $\alpha=0.00047\pm0.00282$, a value that cannot be distinguished from zero [12]. Figure 3(b) shows the sum $\Omega(\epsilon)$, computed using $\Delta\lambda$ values at 2×10^5 uniformly distributed θ points in Fig. 3(a), versus $1/\epsilon$ on a logarithmic scale. The sign-singular scaling behavior is robust in this case, and the cancellation exponent is $\kappa=0.504\pm0.005$. This value of κ is larger than the cancellation exponent computed in Fig. 2(b) for the two-dimensional map (2)–(4). Thus, the degree of the fine-scale cancellation in $\Delta\lambda$ for the coupled circle map lattice (5) is much higher than that for the two-dimensional map (2)–(4), indicating much more significant fine-scale fluctuations of the Lyapunov exponent for the former case [compare Fig. 3(a) to Fig. 2(a)]. The cancellation exponent, therefore, *correctly characterizes and differentiates fine-scale properties of the riddled fractal sets with approximately integer exterior dimensions.*

We have also examined other cases. These include (i) various different system sizes and parameter values of the coupled circle map lattice, (ii) the globally coupled Hénon map lattice, and (iii) the diffusively coupled logistic map lattice. For all riddled fractal sets with approximately zero uncertainty exponents in these systems, the sign-singular scaling behavior gives nonzero cancellation exponents. The existence of nontrivial cancellation exponents in all these systems suggests that sign-singular scaling behavior may be a common feature for riddled fractal sets with near integer exterior dimensions. The fact that different systems exhibit different cancellation exponents provides another way to characterize these exotic fractal sets.

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[3] Grebogi *et al.* [C. Grebogi, S. W. McDonald, E. Ott, and J. A. Yorke, Phys. Lett. A **110**, 1 (1985)] introduced the definition of exterior dimension to characterize fat fractals. Let S be a closed, bounded fat fractal set, and let $S_{(\epsilon)}$ be the set that includes the original set S and all points within a distance ϵ of S. Define the exterior dimension to be $d_{ex}=D-\lim_{\epsilon\to0}\ln V(\bar{S}(\epsilon))/\ln\epsilon$, where $\bar{S}_{(\epsilon)}=S_{(\epsilon)}-S$ is what remains of $S_{(\epsilon)}$ if the original set S is deleted, and V denotes volume. For thin fractal sets, $V(S)=0$ and, hence, d_{ex} reduces to the usual box-counting dimension. For fat fractal sets, d_{ex} is related to the uncertainty exponent α, since $P(\epsilon)$ can also be interpreted as the probability of making an error by saying that a point $\mathbf{x}_0$ lies in S if the perturbed point $\mathbf{x}_0\pm\epsilon$ does not lie in S, or the probability of being wrong by saying that a point $\mathbf{x}_0$ does not lie in S when $\mathbf{x}_0\pm\epsilon$ is determined to lie in S. Thus, $P(\epsilon)$ is proportional to $V(\bar{S}(\epsilon))$. Since $P(\epsilon)\sim\epsilon^{\alpha}$, this leads to $\alpha=D-d_{ex}$. For the logistic map, α is equivalent to another exponent introduced by Farmer [C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **56**, 266 (1986)].

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[10] We choose to study the sign scaling behavior of the maximum Lyapunov exponent because it characterizes the nature of the asymptotic attractors (chaotic versus nonchaotic). Other physical quantities that are defined with respect to the riddled fractal sets, which change sign on fine scales, can exhibit the sign-singular scaling behavior and thus can also be used to characterize the riddled fractals.

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